Explicit Numerical Method for Solution of Heat-Transfer Problems

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Theme

AN explicit numerical method for the solution of heat-transfer problems is presented. The primary equation represents the solution of a first-order linear differential equation for the temperature of cell i in a system of N cells. The basic equation is referred to as the rate-exponential equation to distinguish it from an unconditionally stable exponential equation presented by Larkin in Ref. 1. The major difference between the two is that the rate-exponential equation accounts for temperature rates of those cells exchanging heat with cell i, whereas Larkin's equation does not. More accurate solutions are obtained as a result of this refinement. Unlike the Larkin equation, the rate-exponential equation must satisfy a stability criterion. However, the criterion is less restrictive than the one associated with a highly accurate and commonly used explicit method which is referred to in this article as the conventional method. The proposed method therefore retains a high level of precision when compared with Larkin's method and relaxes the stability criterion when compared with the conventional method.

Contents

Numerical methods are used extensively to solve practical problems in heat transfer.[1–6] These methods usually consist of dividing the system into N cells of finite dimensions and computing the temperature of each cell in time steps of finite duration, based on local values of geometry, thermal property, and temperature distribution. The various methods are often categorized with regard to whether or not the arguments are known prior to making the computation. If all arguments are known, the method is explicit, otherwise the method is implicit. The method detailed here is explicit.

The objective is to solve the transient thermal conduction equation presented below for a rectangular coordinate system

$$\nabla \cdot (k \nabla T) = \rho c (\partial T / \partial \theta) - Q' \tag{1}$$

where T, k, ρ, c, Q', and θ are temperature, thermal conductivity, density, specific heat, heat rate per unit volume, and time, respectively.

Consideration is first given to a commonly used explicit equation, since it and its stability criterion are relevant to the development of the rate-exponential equation and its stability criterion. Equation (1) is approximated by replacing the partial derivatives with difference expressions. Dimensional differences are determined based on the dimensions of each cell of an N cell system, where cell k has the dimensions Δx_k, Δy_k, and Δz_k. Time is divided into intervals of $\Delta\theta$ duration. If the subscript i is assigned to the cell whose temperature is to be computed and the subscript j is assigned to each of J cells in the system which exchange heat with cell i, then Eq. (1) is represented locally at cell i by the following:

$$T_{i,\theta+\Delta\theta} = T_{i,\theta}\left(1 - \frac{\Delta\theta}{C_i}\sum_{j=1}^{J} Y_{i,j}\right)_{\theta} + \frac{\Delta\theta}{C_{i,\theta}}\left(Q_i + \sum_{j=1}^{J} Y_{i,j} T_j\right)_{\theta} \tag{2}$$

where $C(=\rho c v)$ is capacitance, $Y[=(ak)/L]$ is admittance, $Q(=Q'v)$ is heat rate, and where a, L, and v are interface area between cells, length between cell centers, and cell volume, respectively. The subscripts θ and $\theta+\Delta\theta$ denote the beginning and end of the time step, respectively. Further properties of Eq. (2) are that $Y_{i,j} = Y_{j,i}$ and $Y_{i,i} = 0$. Equation (2) is referred to as the conventional equation and, in the form shown, is not necessarily limited to a rectangular coordinate system.

The stability criterion associated with Eq. (2) is the following[3,7–10]:

$$\Delta\theta_{\text{stable}} \leqq \left[C_i \bigg/ \sum_{j=1}^{J} Y_{i,j}\right]_{\text{least of set } N} \tag{3}$$

In this article stability criteria such as Eq. (3) do not apply to instabilities that may result from nonlinearities associated with expressing thermal properties or boundary conditions as functions of temperature. An examination of Eq. (3) shows that cells which represent metals (high k) or gases (low ρc) dictate that a relatively small time step be used in the solution of the conventional equation.

The rate exponential equation is now derived from Eq. (1) by replacing space-related partial derivatives by equivalent differences and replacing the time-related partial derivative by the total derivative. Use of the total derivative with time is justified since the space coordinates of each cell are fixed locally. The following total differential equation therefore applies at cell i, where the subscript θ again denotes that thermal properties are evaluated at the beginning of the time step:

$$\frac{dT_i}{d\theta} = -T_i\left[\frac{1}{C_i}\sum_{j=1}^{J} Y_{i,j}\right]_{\theta} + \frac{1}{C_{i,\theta}}\left(Q_{i,\theta} + \sum_{j=1}^{J} Y_{i,j,\theta} T_j\right) \tag{4}$$

During the current time step, T_j is defined by the following:

$$T_j = T_{j,\theta} + \dot{T}_j \theta' \tag{5}$$

where θ' is time measured from the beginning of the time step. It is noted that the Larkin method assumes $T_j = T_{j,\theta}$. The only significant difference between this and Larkin's derivation occurs here in the definition of T_j.

When Eq. (5) is substituted into Eq. (4), the following is obtained. The independent variable is now θ' since derivatives with respect to θ and θ' are equal

$$(D + \alpha_i) T_i = \beta_i + \gamma_i \theta' \tag{6}$$

where

$$\alpha_i = \left[(1/C_i)\sum_{j=1}^{J} Y_{i,j}\right]_{\theta}, \qquad \beta_i = \left[(1/C_i)\left(Q_i + \sum_{j=1}^{J} Y_{i,j} T_j\right)\right]_{\theta}$$

and

$$\gamma_i = \left[(1/C_i)\sum_{j=1}^{J} Y_{i,j}\dot{T}_j\right]_{\theta}$$

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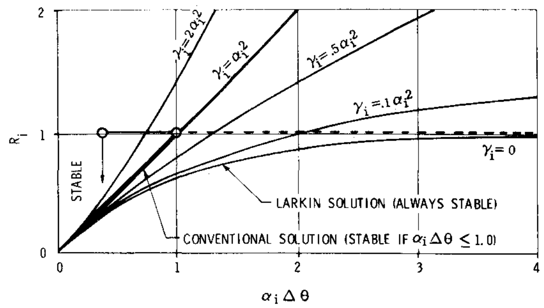

Fig. 1 Parametric solutions of rate-exponential equation.

The solution of Eq. (6) is the following:

$$T_i = C_1 e^{-\alpha_i \theta'} + (\beta/\alpha)_i + (\gamma/\alpha)_i[\theta' - (1/\alpha_i)] \tag{7}$$

To be consistent with previously established nomenclature, let $T_i = T_{i,\theta}$ at the beginning of the time step when $\theta' = 0$, then the constant of integration is the following:

$$C_1 = T_{i,\theta} - (\beta/\alpha)_i + (\gamma/\alpha^2)_i$$

The desired form of the rate-exponential equation is obtained when the value of C_1 is substituted into Eq. (7) and Eq. (7) is evaluated at the end of the time step

$$T_{i,\theta+\Delta\theta} = T_{i,\theta} e^{-\alpha_i \Delta\theta} + \frac{1}{\alpha_i}\left(\beta - \frac{\gamma}{\alpha}\right)_i (1 - e^{-\alpha_i \Delta\theta}) + \left(\frac{\gamma}{\alpha}\right)_i \Delta\theta \tag{8}$$

When Eq. (8) is compared with the Larkin equation (Eq. 10 of Ref. 1), it is seen that the two are identical if $\gamma = 0$ in Eq. (8).

A dimensionless temperature parameter R_i is now defined that will be used to derive a stability criterion for the rate-exponential equation

$$R_i = (T_{\theta+\Delta\theta} - T_\theta)_i/(T_{\theta+\Delta_\theta} - T_\theta)_{i,\ \gamma=0,\alpha_i\Delta\theta\to\infty,\text{normalized}} \tag{9}$$

To express R_i in terms of the parameters α_i, β_i, and γ_i, it is necessary to substitute $T_{i,\theta+\Delta\theta}$ from Eq. (8) into Eq. (9) and set the denominator of Eq. (9), which equals $[(\beta/\alpha)_i - T_{i,\theta}]$, to 1.0. The following result is then obtained:

$$R_i = [1 - (\gamma/\alpha^2)_i](1 - e^{-\alpha_i \Delta\theta}) + (\gamma/\alpha)_i \Delta\theta \tag{10}$$

Figure 1 shows a parametric plot of R_i with its values for the conventional equation superimposed. Note that the conventional equation is stable if $R_i \leq 1.0$. Also note that for Larkin's equation, which is unconditionally stable, R_i is always less than one. It is therefore inferred that the stability criterion for the rate-exponential equation is the following:

$$R_{\text{stable}} = R_{i,\text{largest of set } N} \leqq 1.0 \tag{11}$$

A detailed examination of computed values has shown this inference to be valid. In many cases the dynamics of the system permitted stable solutions to be obtained while $R_{i,\max}$ exceeded unity.

An existing computer program, which solves heat-transfer problems by using the conventional equation, was modified so that solutions could be obtained by using the rate-exponential and Larkin equations as well. To evaluate the accuracy and stability of these three methods, a problem for which an exact solution exists was solved by each method. The numerical method solutions were then compared with the exact solution. The problem selected for this comparison was that of determining the temperature of a slab exposed to a constant heat flux at the front face. The exact solution of this problem is discussed in Sec. 3.8 of Ref. 2. The results of the comparison are shown in Fig. 2. The Larkin method clearly underpredicted temperatures, the magnitude of the error increasing with increasing $\Delta\theta$. On the other hand, the conventional and rate-exponential methods predicted temperatures very accurately. It is apparent from Fig. 1 that the Larkin equation ($\gamma_i = 0$) will always underpredict temperature changes, as it did in this example, as long as

$$\sum_{j=1}^{J} Y_{i,j} \dot{T}_j \neq 0.$$

This error, however, will be minimized for problems in which the parameter $\alpha_i \Delta\theta$ is very small, since all solutions converge as $\alpha_i \Delta\theta$ approaches zero.

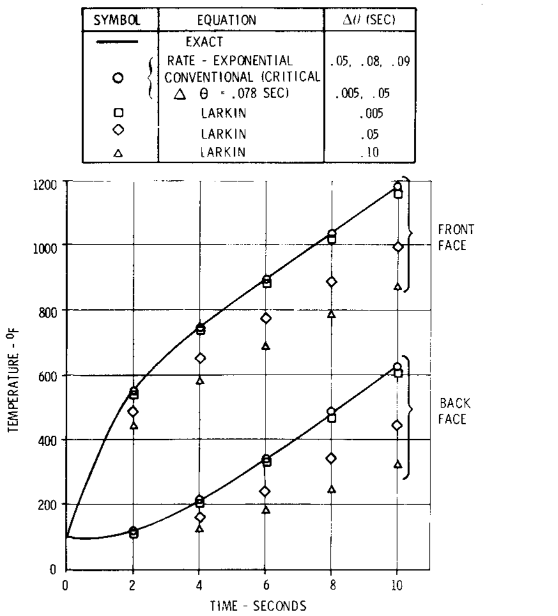

Fig. 2 Slab temperature: heat flux = 150 Btu/ft²-sec; slab thickness = 0.04167 ft; thermal conductivity = 20 Btu/ft-hr-°F; density × specific heat = 50 Btu/ft³-°F; 11 cells/model.

As shown in Fig. 2, accurate solutions were obtained with the rate-exponential equation for time steps up to and including 0.09 sec. This is approximately a 15% larger time step than the time step permitted for the conventional equation (0.078 sec). In general, the stability criterion for the rate-exponential equation will permit a larger time step than will the stability criterion for the conventional equation because, for most practical problems, $\gamma_i < \alpha_i^2$. The efficiency of the rate-exponential solution can be improved if the maximum stable time step is computed and used during each step of the integration.

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